An Unsplit Formulation of the Berenger's PML Absorbing Boundary Condition for FDTD Meshes

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***Abstract*—An unsplit formulation of the Berenger's perfectly matched layer absorbing boundary condition (PML ABC) for finite-difference time-domain (FDTD) meshes is presented. This unsplit formulation uses the conventional E-H algorithm, but does not require the E and H fields to be split. The proposed formulation is memory-efficient like that of the previous unsplit PML following the theory of Sacks. This unsplit formulation is easy to be implemented, and many useful modification to the Berenger's PML can also be done to it.**

***Index Terms*—Absorbing boundary condition, finite-difference time-domain method, perfectly matched layer.**

I. Introduction

BERENGER'S perfectly matched layer (PML) absorbing boundary condition (ABC) [1] was one of the most important developments in the use of the finite-difference time-domain (FDTD) method. Berenger's PML requires the E and H fields to be split, this increases the memory requirements of the FDTD procedure. The successful implementation of unsplit PML following the theory of Sacks has been done by ZHAO [2] and Sullivan [3]. But many useful modification(such as the modification in MPML [4]) done to the Berenger's PML can not be used with this kind of unsplit PML.

In this letter, an unsplit formulation of the Berenger's PML is introduced. This unsplit formulation uses the conventional E-H algorithm, but does not require the E and H fields to be split. Like original Berenger's PML, many useful modification to the Berenger's PML can also be used with this formulation. Examples show the efficiency of the proposed unsplit formulation.

II. Fomulation

Consider a PML with $\sigma_y = \sigma_y^* = \sigma_z = \sigma_z^* = 0$, the original Berenger's PML results in 12 split field eqn [5] in 3-D, as follows:

$$\mu_0 \frac{\partial H_{xy}}{\partial t} = -\frac{\partial E_z}{\partial y} \tag{1a}$$

$$\mu_0 \frac{\partial H_{xz}}{\partial t} = \frac{\partial E_y}{\partial z} \tag{1b}$$

$$\mu_0 \frac{\partial H_{yz}}{\partial t} = -\frac{\partial E_x}{\partial z} \tag{1c}$$

$$\mu_0 \frac{\partial H_{yx}}{\partial t} + \sigma_x^* H_{yx} = \frac{\partial E_z}{\partial x} \tag{1d}$$

$$\mu_0 \frac{\partial H_{zx}}{\partial t} + \sigma_x^* H_{zx} = -\frac{\partial E_y}{\partial x} \tag{1e}$$

$$\mu_0 \frac{\partial H_{zy}}{\partial t} = \frac{\partial E_x}{\partial y} \tag{1f}$$

$$\varepsilon_0 \frac{\partial E_{xy}}{\partial t} = \frac{\partial H_z}{\partial y} \tag{2a}$$

$$\varepsilon_0 \frac{\partial E_{xz}}{\partial t} = -\frac{\partial H_y}{\partial z} \tag{2b}$$

$$\varepsilon_0 \frac{\partial E_{yz}}{\partial t} = \frac{\partial H_x}{\partial z} \tag{2c}$$

$$\varepsilon_0 \frac{\partial E_{yx}}{\partial t} + \sigma_x E_{yx} = -\frac{\partial H_z}{\partial x} \tag{2d}$$

$$\varepsilon_0 \frac{\partial E_{zx}}{\partial t} + \sigma_x E_{zx} = \frac{\partial H_y}{\partial x} \tag{2e}$$

$$\varepsilon_0 \frac{\partial E_{zy}}{\partial t} = -\frac{\partial H_x}{\partial y}. \tag{2f}$$

From (1a) and (1b), we can get an unsplit field component equation for H_x

$$\mu_0 \frac{\partial H_x}{\partial t} = -\frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z}. \tag{3}$$

Using time step Δt, and using the usual first-order difference approximations in time and space, the following difference equation for (3) is obtained:

$$\begin{aligned} H_x^{n+1/2}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right) =& H_x^{n-1/2}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right) \\ &+ \frac{\Delta t}{\mu_0 \cdot \Delta z}\left[E_y^n\left(i, j+\frac{1}{2}, k+1\right) - E_y^n\left(i, j+\frac{1}{2}, k\right)\right] \\ &- \frac{\Delta t}{\mu_0 \cdot \Delta y}\left[E_z^n\left(i, j+1, k+\frac{1}{2}\right) - E_z^n\left(i, j, k+\frac{1}{2}\right)\right]. \end{aligned} \tag{4}$$

Difference equation for (1c) is

$$H_{yz}^{n+1/2}\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right) = H_{yz}^{n-1/2}\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right) - \frac{\Delta t}{\mu_0 \cdot \Delta z}\left[E_x^n\left(i+\frac{1}{2}, j, k+1\right) - E_x^n\left(i+\frac{1}{2}, j, k\right)\right]. \tag{5}$$

Manuscript received September 26, 2002; revised February 20, 2003. This work was supported by the National Science Foundation of China under Grant 60172002. The review of this letter was arranged by Associate Editor Dr. Rüdiger Vahldieck.

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Digital Object Identifier 10.1109/LMWC.2003.815694

Defining the integral of E_x as

$$sumE_x^{n-1}\left(i+\frac{1}{2},j,k\right) = \sum_{i=1}^{n} E_x^{i-1}\left(i+\frac{1}{2},j,k\right) = sumE_x^{n-2}\left(i+\frac{1}{2},j,k\right) + E_x^{n-1}\left(i+\frac{1}{2},j,k\right). \quad (6)$$

Then using recursive method in eqn (5), and assuming that initially at time $t=0$ ($n=0$), all the field components are zero over the whole computational domain, we have

$$H_{yz}^{n-1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) = -\frac{\Delta t}{\mu_0\cdot\Delta z}\left[sumE_x^{n-1}\cdot\left(i+\frac{1}{2},j,k+1\right) - sumE_x^{n-1}\left(i+\frac{1}{2},j,k\right)\right]. \quad (7)$$

According to [1], [5], the FDTD formulation for (1d) is

$$H_{yx}^{n+1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) = e^{-\sigma_x^*\Delta t/\mu_0} H_{yx}^{n-1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) + \frac{1}{\sigma_x^*\cdot\Delta x}\left(1-e^{-\sigma_x^*\Delta t/\mu_0}\right)\cdot\left[E_z^n\left(i+1,j,k+\frac{1}{2}\right) - E_z^n\left(i,j,k+\frac{1}{2}\right)\right]. \quad (8)$$

$H_y^{n-1/2}(i+1/2,j,k+1/2)$ can be expressed as

$$H_y^{n+1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) = H_{yz}^{n+1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) + H_{yx}^{n+1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right). \quad (9)$$

Substituting (5) and (8) into (9), and eliminating $H_y^{n-1/2}(i+1/2,j,k+1/2)$ with eqn (7), We get the unsplit field component formulation for $H_y^{n+1/2}(i+1/2,j,k+1/2)$

$$\begin{aligned} H_y^{n+1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) &= e^{-\sigma_x^*\Delta t/\mu_0}\left\{H_y^{n-1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)+SA\right\} \\ &\quad - SA + \frac{1}{\sigma_x^*\cdot\Delta x}\left(1-e^{-\sigma_x^*\Delta t/\mu_0}\right) \cdot\left[E_z^n\left(i+1,j,k+\frac{1}{2}\right)-E_z^n\left(i,j,k+\frac{1}{2}\right)\right] \\ &\quad - \frac{\Delta t}{\mu_0\cdot\Delta z}\left[E_x^n\left(i+\frac{1}{2},j,k+1\right) - E_x^n\left(i+\frac{1}{2},j,k\right)\right] \end{aligned} \quad (10)$$

where

$$SA = \frac{\Delta t}{\mu_0\cdot\Delta z}\left[sumE_x^{n-1}\left(i+\frac{1}{2},j,k+1\right) - sumE_x^{n-1}\left(i+\frac{1}{2},j,k\right)\right]. \quad (11)$$

$$\begin{aligned} H_z^{n+1/2}\left(i+\frac{1}{2},j+\frac{1}{2},k\right) =& e^{-\sigma_x^*\Delta t/\mu_0}\left[H_z^{n-1/2}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)-SB\right]+SB \\ &- \frac{1}{\sigma_x^*\cdot\Delta x}\left(1-e^{-\sigma_x^*\Delta t/\mu_0}\right)\left[E_z^n\left(i+1,j,k+\frac{1}{2}\right)-E_z^n\left(i,j,k+\frac{1}{2}\right)\right] \\ &+ \frac{\Delta t}{\mu_0\cdot\Delta z}\left[E_x^n\left(i+\frac{1}{2},j,k+1\right)-E_x^n\left(i+\frac{1}{2},j,k\right)\right] \end{aligned} \quad (12)$$

$$\begin{aligned} E_x^{n+1}\left(i+\frac{1}{2},j,k\right) =& E_x^n\left(i+\frac{1}{2},j,k\right) \\ &+ \frac{\Delta t}{\varepsilon_0\cdot\Delta y}\left[H_z^{n+1/2}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)-H_z^{n+1/2}\left(i+\frac{1}{2},j-\frac{1}{2},k\right)\right] \\ &- \frac{\Delta t}{\varepsilon_0\cdot\Delta z}\left[H_y^{n+1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)-H_y^{n+1/2}\left(i+\frac{1}{2},j,k-\frac{1}{2}\right)\right] \end{aligned} \quad (13)$$

$$\begin{aligned} E_y^{n+1}\left(i,j+\frac{1}{2},k\right) =& e^{-\sigma_x\Delta t/\varepsilon_0}\left[E_y^n\left(i,j+\frac{1}{2},k\right)-SC\right]+SC \\ &- \frac{1}{\sigma_x\cdot\Delta x}\left(1-e^{-\sigma_x\Delta t/\varepsilon_0}\right)\left[H_z^{n+1/2}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)-H_z^{n+1/2}\left(i-\frac{1}{2},j+\frac{1}{2},k\right)\right] \\ &+ \frac{\Delta t}{\varepsilon_0\cdot\Delta z}\left[H_x^{n+1/2}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)-H_x^{n+1/2}\left(i,j+\frac{1}{2},k-\frac{1}{2}\right)\right] \end{aligned} \quad (14)$$

$$\begin{aligned} E_z^{n+1}\left(i,j,k+\frac{1}{2}\right) =& e^{-\sigma_x\Delta t/\varepsilon_0}\left[E_z^n\left(i,j,k+\frac{1}{2}\right)+SD\right]-SD \\ &+ \frac{1}{\sigma_x\cdot\Delta x}\left(1-e^{-\sigma_x\Delta t/\varepsilon_0}\right)\left[H_y^{n+1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)-H_y^{n+1/2}\left(i-\frac{1}{2},j,k+\frac{1}{2}\right)\right] \\ &- \frac{\Delta t}{\varepsilon_0\cdot\Delta y}\left[H_x^{n+1/2}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)-H_x^{n+1/2}\left(i,j-\frac{1}{2},k+\frac{1}{2}\right)\right] \end{aligned} \quad (15)$$

Similarly, we can have other unsplit field component formulation (see (12)–(15) at the bottom of the previous page) where

$$SB = \frac{\Delta t}{\mu_0 \cdot \Delta y}\left[sumE_x^{n-1}\left(i+\frac{1}{2}, j+1, k\right) - sumE_x^{n-1}\left(i+\frac{1}{2}, j, k\right)\right] \quad (16)$$

$$SC = \frac{\Delta t}{\varepsilon_0 \cdot \Delta z}\left[sumH_x^{n-1/2}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right) - sumH_x^{n-1/2}\left(i, j+\frac{1}{2}, k-\frac{1}{2}\right)\right] \quad (17)$$

$$SD = \frac{\Delta t}{\varepsilon_0 \cdot \Delta y}\left[sumH_x^{n-1/2}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right) - sumH_x^{n-1/2}\left(i, j-\frac{1}{2}, k+\frac{1}{2}\right)\right] \quad (18)$$

$$\begin{aligned} sumH_x^{n-1/2}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right) &= \sum_{i=1}^{n} H_x^{i-1/2}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right) \\ &= sumH_x^{n-3/2}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right) + H_x^{n-1/2}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right). \end{aligned} \quad (19)$$

From the derived unsplit field component equations above, we can see that only two quantities, the integral of E_x and integral of H_x have to be stored, in place of four quantities, E_{yz}, E_{zy}, H_{yz}, H_{zy}, with initial implementation.

III. Numerical Validation

In order to validate numerically the derived unsplit formulation of Berenger's PML, we conducted a numerical experiment in a uniform rectangular cross-section waveguide. The dimension of the cross section is 0.41 m $\times$ 0.2 m in the x and y derections respectively, and the distance between two ends is 0.2 m. A uniform mesh with $\Delta x = \Delta y = \Delta z = 0.01$ m was used, so the entire computational domain Ω_N includes $41 \times 20 \times 20$ cells. One end of the waveguide is terminated by perfectly electric conductors, other end is terminated by one of 8-cell-thick Berenger's PML, proposed unspilt PML, MPML, and unsplit MPML (i.e., MPML using the proposed unsplit formulation), respectively, also backed by perfectly electric conductors. A $\lambda/2$ dipole with 0.001 m diameter, fed by a sinusoidal wave with a frequency of 1000 MHz is located along x direction 2-cells away from the PML in the waveguide. The dipole and the cross-section containing it have the same center. The benchmark FDTD solution ,with zero truncation boundary reflections, was obtained by running a large mesh in Ω_L which the truncation boundary was placed sufficiently far away to provide for causal isolation for all points in Ω_N over the time interval used for comparisons.

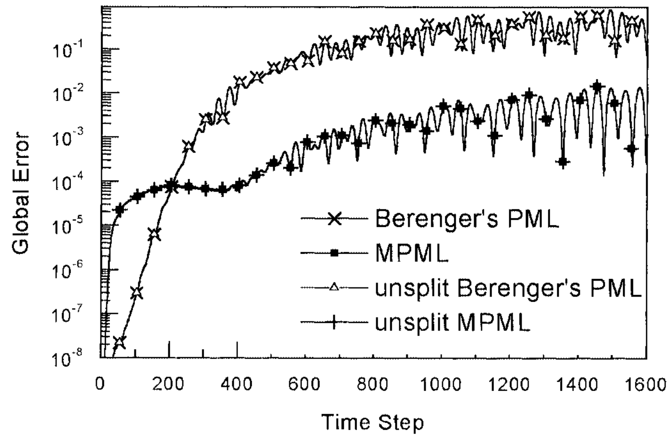

Fig. 1. Global error energy (square of the electric field error at each grid cell summed throughout the entire grid) within the $41 \times 20 \times 20$ cell FDTD grid for four kinds of 8-cell-thick PML, plotted as a function of time step number on a logarithmic vertical scale.

The error due to numerical reflections caused by the presence of the conductor-backed PML was obtained by subtracting the field at any point inside Ω_N from the field at the corresponding point in Ω_L. For the numerical experiment maximum conductivity and order of spatial polynomial are taken as $\sigma_{\max} = 0.457\,98$ S/m (i.e., $R = 0.0001$) and $n = 2$, respectively. Fig. 1 graphs the global error energy for four cases mentioned above. It can be clearly seen that the global error energy for 8-cell-thick Berenger's PML and 8-cell-thick unsplit PML are the same, and the global error energy for 8-cell-thick MPML and 8-cell-thick unsplit MPML are the same too. After $n = 400$ time steps, the global error energy for 8-cell-thick unsplit MPML is about 10^{-2} that of 8-cell-thick Berenger's PML.

IV. Conclution

In conclusion, an unsplit formulation of the Berenger's PML ABC has been presented. The proposed unsplit formulation not only maintains the efficient ability of using modifications done to the original Berenger's PML, but has the advantage of reducing the memory requirements.

References

[1] J. P. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *J. Comput. Phys.*, vol. 114, pp. 185–200, 1994.

[2] L. Zhao and A. C. Cangellaris, "A general approach for the development of unsplit-field time-domain implementations of perfectly matched layers for FDTD grid truncation," *IEEE Microwave Guided Wave Lett.*, vol. 6, pp. 209–211, May 1996.

[3] D. M. Sullivan, "An unsplit step 3-D PML for use with the FDTD method," *IEEE Microwave Guided Wave Lett.*, vol. 7, pp. 184–186, Nov. 1997.

[4] B. Chen, D. G. Fang, and B. H. Zhou, "Modified berenger PML absorbing boundary condition for FD-TD meshes," *IEEE Microwave Guided Wave Lett.*, vol. 5, pp. 399–401, Nov. 1995.

[5] D. S. Katz, E. T. Thiele, and A. Taflove, "Validation and extension to three dimensional of the berenger PML absorbing boundary condition for FDTD meshes," *IEEE Microwave Guided Wave Lett.*, vol. 4, pp. 268–270, Aug. 1994.